

Mathematical Properties Of American Chooser Options

Shi Qiu¹ and Sovan Mitra^{2*}

The American chooser option is a relatively new compound option that has the characteristic of offering exceptional risk reduction for highly volatile assets. This has become particularly significant since the start of the global financial crisis. In this paper we derive mathematical properties of American chooser options. We show that the two optimal stopping boundaries for American chooser options with finite horizon can be characterised as the unique solution pair to a system formed by two nonlinear integral equations, arising from the early exercise premium representation. The proof of early exercise premium representation is based on the method of change-of-variable formula with local time on curves. The key mathematical properties of American chooser options are proved, specifically smooth-fit, continuity of value function and continuity of free-boundary amongst others. We compare the performance of the American chooser option against the American strangle option. We also conduct numerical experiments to illustrate our results.

Keywords: American chooser options; American strangle options; change of variable; optimal stopping; free boundaries.

¹ School of Mathematics,
University of Manchester,
Oxford Road,
Manchester,
M13 9PL,
United Kingdom.

² University of Liverpool,
Brownlow Hill,
Liverpool,
L69 7ZX,
United Kingdom.

*corresponding author

1. Introduction

Options and derivatives have played an increasingly important role in risk management (see for instance Deng and Oren (2006), Hull and White (2017) and Wang et al. (2015)), as well as providing useful opportunities for investment. In particular, the commencement of the global financial crisis has emphasised the relevance of hedging instruments such as options as methods of risk management (see for instance Cornett et al. (2011), Aebi et al. (2012)). This also has significant societal impact because many social welfare funds (such as pensions, life insurance etc.) have substantially suffered since the start of the global financial crisis, due to insufficient risk management.

The chooser option is a compound option (see Shiryaev (1999)), that is it is an option on an option. The derivative's payoff depends on the value of another option (see Shreve (2004)). The chooser option gives the right for option holders to choose a call or put option before the choosing maturity date T_1 . Consequently, in a highly volatile market, the chooser option gives the investor a time period to observe the volatility of the underlying asset and select either a call or put option accordingly. Therefore the chooser option is extremely beneficial in highly volatile markets without the risk of vanilla options, which only benefit from volatile movements in a single direction. Hence chooser options provide a unique investment and risk management tool compared to other financial derivatives.

The payoff of American chooser options is the maximum of the value of an American call option and an American put option. As is well known (see for instance Kim (1990) and Jacka (1991)), the value of American options can be expressed through the EEP (early exercise premium) representation and this can be obtained in different ways. The martingale method (see Detemple (2006)) gives the EEP representation for all American type contingent claims and Detemple and Emmerling (2009) use that to price American chooser options. Another method in Alobaidi and Mallier (2002) and Chiarella and Ziogas (2005) apply the Laplace or Fourier transform to partial differential equations to obtain the EEP representation.

In this paper, we use the change-of-variable formula with local time on curves (see Peskir (2005a)) to price the American chooser option. This method has been used in previous research on American options, such as in Peskir (2005b), Peskir (2005c), and Qiu (2014) amongst others. The EEP representation leads to a problem in determining the free boundaries of chooser options. By using the boundary conditions, we form a system of two recursive integral equations and the graph of the free boundaries can be calculated by the numerical method in Qiu (2014). Additionally, we show that the solution pair of the system exists and is unique. We also derive properties of American chooser options, and analyse these properties. Using the skeleton analysis from Peskir and Samee (2011) we examine the performance of options returns.

The paper is organised as follows. In section 2 we give an introduction, the preliminaries and fundamentals of American chooser options. In Section 3 we analyse the optimal stopping region, proving two new Theorems. In Section 4 we examine the mathematical properties of American chooser options, deriving new relationships on the value function (such as convexity and smooth-fit) and the free boundaries. In Section 5 we examine the EEP representation of American chooser options and prove new Theorems with respect to Martingale processes and optimal stopping boundaries. In Section 6 we compare the American chooser option to American strangle options, deriving a new Theorem on the returns of American chooser options compared to American strangle options. In the next section we conduct some numerical experiments to

illustrate our results, and we then end with a conclusion.

2. Preliminaries

Consider the financial market consisting of a risky stock X and a riskless bond B whose prices respectively evolve as (see Wilmott (1998))

$$(2.1) \quad dX_t = (\mu - \delta)X_t dt + \sigma X_t dW_t \quad (X_0 = x),$$

$$(2.2) \quad dB_t = rB_t dt \quad (B_0 = 1),$$

where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion, parameter $\mu \in \mathbb{R}_+$ is the drift, parameter $\sigma \in \mathbb{R}_+$ is the volatility and $\delta \in \mathbb{R}_+$ is the continuous dividend yield. The American chooser option can be treated as an American put option with an opportunity to change it into American call options. From the analysis of American put options in Peskir (2005b), the price of options is determined under the risk neutral measure \tilde{P} , under this measure the stock price in (2.1) will be expressed as

$$(2.3) \quad dX_t = (r - \delta)X_t dt + \sigma X_t dW_t \quad (X_0 = x).$$

For finite expiry American chooser options, the holder can decide to choose to hold an American call or American put option until expiry date T_1 . The holder's option will then expire at expiration date T_2 , such that $T_2 > T_1$. This paper assumes that the maturity for American call options is T_{call} and the maturity for American put options is T_{put} , such that $\min(T_{call}, T_{put}) > T_1$. We set $T_2 = \max(T_{call}, T_{put})$. We note that the European chooser option gives the right to choose either European call or European put option only at the maturity time T_1 . The value of European chooser options can be calculated by the put-call parity in Rubinstein (1991). In Durica and Svabova (2014) there is an implicit formula for the value, Delta and Gamma of European chooser options. The analysis of chooser options of American type began in Detemple and Emmerling (2009), which gives a rigorous proof of the properties of American chooser options.

The perpetual American chooser options (where $T_1 \rightarrow \infty$) has a closed form solution for its value function (see Gapeev and Rodosthenous (2010)). The value of perpetual American chooser option V^{CH} is determined as the optimal stopping time τ to maximise the predicted payoff in the future:

$$(2.4) \quad V^{CH}(x) = \sup_{\tau} E_x(e^{-r\tau} G^{CH}(X_{\tau})),$$

the supremum is taken over all the stopping time of $(X_t)_{t \geq 0}$. Additionally, $G^{CH}(x)$ can be written as

$$(2.5) \quad G^{CH}(x) = \max\{V^C(x), V^P(x)\},$$

where $V^C(x)$ is the value of perpetual American call options with strike price K , and $V^P(x)$ is the value of perpetual American put options with strike price L . A discussion of American chooser options is given in Gapeev and Rodosthenous (2010). With the proof that the solution pair of the free-boundary problem is unique, we obtain the two free boundaries $p_* \in (0, L)$ and

$q_* \in (K, \infty)$, and the value function is given in Proposition 6 of Gapeev and Rodosthenous (2010). In our paper, we mainly focus on the American chooser options with finite horizon.

In the finite horizon case, the value function of American chooser options depends on time and space. Moreover, the free boundaries are not just constant but functions depending on time t . We will now discuss the properties of value functions and the free boundaries. From Detemple and Emmerling (2009) the arbitrage-free price of American chooser options with finite horizon is defined as

$$(2.6) \quad V^{CH}(t, x) = \sup_{0 \leq \tau \leq T_1} \tilde{E}_{t,x}(e^{-r\tau} G^{CH}(t + \tau, X_{t+\tau})),$$

where $\tau \in [0, T_1]$ is the stopping time of the geometric Brownian motion $X(\mu) = (X_{t+s})_{s \geq 0}$ satisfying equation (2.1). The notation $\tilde{E}_{t,x}$ is the expectation under the risk neutral measure $\tilde{P}_{t,x}$ (i.e. $\tilde{P}_{t,x}(X_t = x) = 1$). The payoff function is given by

$$(2.7) \quad G^{CH}(t, x) = \max\{V^P(t, x), V^C(t, x)\},$$

with the value of American call options $V^C(t, x)$ with strike price K , and the value of American put options $V^P(t, x)$ (see Peskir (2005b)) with strike price L . When $L > K$, we call the chooser option an American complex chooser option. The maturity of $V^C(t, x)$ and $V^P(t, x)$ are T_{call} and T_{put} , respectively. We set $T_2 = \min(T_{call}, T_{put})$, and T_1 is the choosing maturity satisfying $T_1 < T_2$. The value function (2.6) can be written as

$$(2.8) \quad V^{CH}(t, x) = \sup_{0 \leq \tau \leq T_1} E_{t,x}(e^{-r\tau} G^{CH}(t + \tau, X_{t+\tau})),$$

for $t \in [0, T_1]$ and $x \in (0, \infty)$ where the supremum is taken as in (2.6) and the process $X = X(r)$ in (2.8) under P solves

$$(2.9) \quad dX_{t+s} = (r - \delta)X_{t+s} ds + \sigma X_{t+s} dW_s \quad (X_0 = x).$$

The stochastic process $X = X(r)$ is a strong Markov process and its infinitesimal generator is

$$(2.10) \quad \mathbb{L}_X = (r - \delta)x \frac{\partial}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2}.$$

The continuation region and stopping region of the call option $V^C(t, x)$ are

$$(2.11) \quad \begin{aligned} C^C &= \{(t, x) \in [0, T_{call}] \times (0, \infty) | V^C(t, x) > (x - K)^+\} \\ &= \{(t, x) \in [0, T_{call}] \times (0, \infty) | x < b^C(t)\}, \end{aligned}$$

$$(2.12) \quad \begin{aligned} \bar{D}^C &= \{(t, x) \in [0, T_{call}] \times (0, \infty) | V^C(t, x) = (x - K)^+\} \\ &= \{(t, x) \in [0, T_{call}] \times (0, \infty) | x \geq b^C(t)\}, \end{aligned}$$

with free boundary $b^C(t) > K$ for $t \in [0, T_{call}]$. The continuation region and optimal stopping region of the put option $V^P(t, x)$ are

$$(2.13) \quad \begin{aligned} C^P &= \{(t, x) \in [0, T_{put}] \times (0, \infty) | V^P(t, x) > (L - x)^+\} \\ &= \{(t, x) \in [0, T_{put}] \times (0, \infty) | x > b^P(t)\}, \end{aligned}$$

$$(2.14) \quad \begin{aligned} \bar{D}^P &= \{(t, x) \in [0, T_{put}] \times (0, \infty) | V^P(t, x) = (L - x)^+\} \\ &= \{(t, x) \in [0, T_{put}] \times (0, \infty) | x \leq b^P(t)\}, \end{aligned}$$

with free boundary $b^P(t) < L$ for $t \in [0, T_{put}]$.

3. Optimal Stopping Region

Following on from previous work on optimal stopping problems (see Peskir and Shiryaev (2005)), we assert that the continuation region of American chooser options equals

$$(3.1) \quad C^{CH} = \{(t, x) \in [0, T_1] \times (0, \infty) | V^{CH}(t, x) > G^{CH}(t, x)\},$$

and the stopping region equals

$$(3.2) \quad \bar{D}^{CH} = \{(t, x) \in [0, T_1] \times (0, \infty) | V^{CH}(t, x) = G^{CH}(t, x)\}.$$

Since $(t, x) \mapsto G^{CH}(t, x)$ is a continuous function, function $(t, x) \mapsto V^{CH}(t, x)$ is lower semi-continuous by the Statement (2.2.80) in Peskir and Shiryaev (2005). We can then apply Corollary 2.9 in Peskir and Shiryaev (2005), and the optimal stopping time for problem (2.8) is

$$(3.3) \quad \tau_{\bar{D}} = \inf\{0 \leq s \leq T - t | X_{t+s} \in \bar{D}^{CH}\}.$$

If we examine the payoff function G^{CH} defined in (2.7), the value of American call options $x \mapsto V^C(t, x)$ is the strictly increasing function with $V^C(t, 0) < V^P(t, 0)$ and $V^C(t, \infty) = \infty$. The value function of American put options $x \mapsto V^P(t, x)$ is strictly decreasing function with $V^P(t, 0) = L$ and $V^P(t, \infty) < V^C(t, \infty)$. For t given and fixed, there must exist a unique intersection $g(t)$ between $x \mapsto V^C(t, x)$ and $x \mapsto V^P(t, x)$ (see Figure 1). By the Implicit Function Theorem, the function $t \mapsto g(t)$ is continuous and the first derivative exists. So $V^C(t, g(t)) = V^P(t, g(t))$, $V^C(t, x) > V^P(t, x)$ for $x > g(t)$, and $V^C(t, x) < V^P(t, x)$ for $x < g(t)$.

We now state our Theorem on the continuation region C^{CH} and stopping region \bar{D}^{CH} .

Theorem 1. *For $t \leq T_1$, if the point $(t, x) \in C^C \cap C^P$ then this point $(t, x) \in C^{CH}$, and the point $(t, g(t))$ are inside the continuation region C^{CH} . Moreover, the optimal stopping region \bar{D}^{CH} defined in (3.2) can be separated into two disjoint regions*

$$(3.4) \quad \bar{D}_1^{CH} = \{(t, x) \in [0, T] \times (0, \infty) | V^{CH}(t, x) = V^P(t, x)\},$$

$$(3.5) \quad \bar{D}_2^{CH} = \{(t, x) \in [0, T] \times (0, \infty) | V^{CH}(t, x) = V^C(t, x)\}.$$

where $\bar{D}_1^{CH} \cap \bar{D}_2^{CH} = \emptyset$, $\bar{D}_1^{CH} \cup \bar{D}_2^{CH} = \bar{D}^{CH}$. Also, the regions \bar{D}_1^{CH} and \bar{D}_2^{CH} satisfy

- (i) For any t and $x > \max(q_*, \tilde{a})$, the point $(t, x) \in \bar{D}_2^{CH}$. The value q_* is the upper free boundary for perpetual American chooser options (see Gapeev and Rodosthenous (2010)) and \tilde{a} is the free boundary for perpetual American call options (see Gapeev and Lerche (2011)).
- (ii) For any t and $x < \min(p_*, \tilde{b})$, the point $(t, x) \in \bar{D}_1^{CH}$. The value p_* is the lower free boundary for perpetual American chooser options (see Gapeev and Rodosthenous (2010)) and \tilde{b} is the free boundary for perpetual American put options (see Gapeev and Lerche (2011)).
- (iii) Up-connectedness: if $(t, x) \in \bar{D}_2^{CH}$, then the point $(t, \lambda x) \in \bar{D}_2^{CH}$ for all $\lambda \geq 1$.
- (iv) Down-connectedness: if $(t, x) \in \bar{D}_1^{CH}$, then the point $(t, \lambda x) \in \bar{D}_1^{CH}$ for all $\lambda \leq 1$.

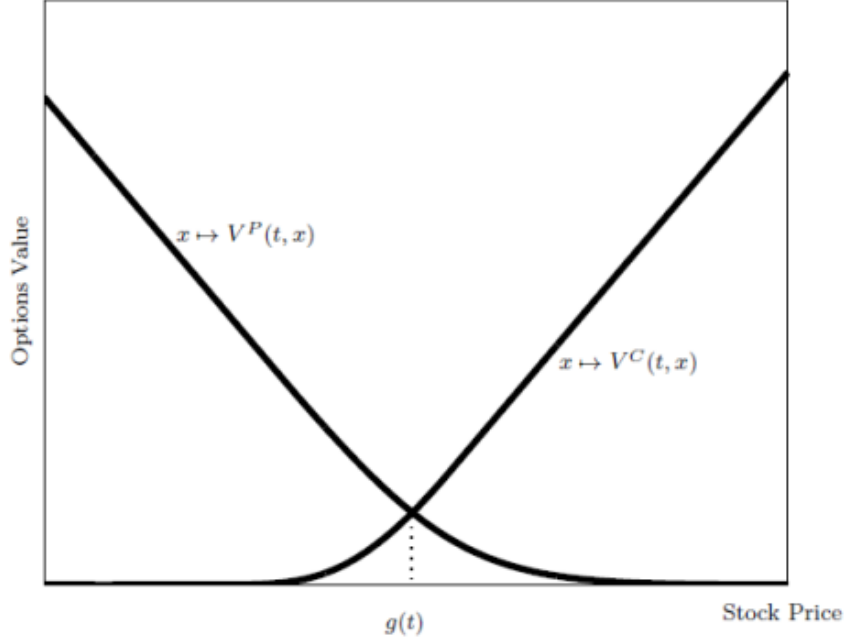


Figure 1. For t given and fixed, the value of American call options and the value of American put options. The point $g(t)$ is the unique intersection between function $x \mapsto V^C(t, x)$ and function $x \mapsto V^P(t, x)$.

Proof. For $t \leq T_1$, if the point $(t, x) \in C^C \cap C^P$ then this point $(t, x) \in C^{CH}$ is proved by the assumption of no arbitrage (see Proposition 1 in Detemple and Emmerling (2009)). We also note that if immediate exercise is suboptimal for both American call and put options, then it is also suboptimal for American chooser options. For $t \leq T_1$, the point $(t, g(t))$ is inside the continuation C^{CH} is proven as follows. If $L \leq K$ then from our previous proof we can separate the optimal stopping region into two parts. By the assumption of no-arbitrage this property can be proven; the details are in (v) of Proposition 2 in Detemple and Emmerling (2009). When $L > K$ (the complex chooser option), the proof of the Theorem is also shown in (v) of Proposition 7 in Detemple and Emmerling (2009). Consequently, the graph of $t \mapsto g(t)$ is inside the continuation region of chooser options. So the optimal stopping \bar{D}^{CH} defined in (3.2) can be separated into two disjoint regions

$$(3.6) \quad \bar{D}_1^{CH} = \{(t, x) \in [0, T] \times (0, \infty) | V^{CH}(t, x) = V^P(t, x)\},$$

$$(3.7) \quad \bar{D}_2^{CH} = \{(t, x) \in [0, T] \times (0, \infty) | V^{CH}(t, x) = V^C(t, x)\}.$$

where $\bar{D}_1^{CH} \cap \bar{D}_2^{CH} = \emptyset$ and $\bar{D}_1^{CH} \cup \bar{D}_2^{CH} = \bar{D}^{CH}$.

The remainder of the theorem is proved as follows: (i). For $x > \max(q_*, \tilde{a})$, we know that $V^{CH}(x) = V^C(x) = (x - K)^+$, where $V^{CH}(x)$ is the value of perpetual American chooser options and $V^C(x)$ is the value of perpetual American call options. Since we know that the value of American chooser options with finite horizon is smaller than the perpetual American chooser options, then $V^{CH}(t, x) \leq V^{CH}(x) = (x - K)^+$. On the other hand, we know that $V^{CH}(t, x) \geq V^C(t, x) \geq (x - K)^+$. So then we have $V^{CH}(t, x) = (x - K)^+ = V^C(t, x)$, i.e. $(t, x) \in \bar{D}_2^{CH}$. Similarly, we can prove (ii). The proof of statements (iii) and (iv) can be found

in (v) of Proposition 2 in Detemple and Emmerling (2009). □

Currently, we know that the region \bar{D}_2^{CH} is non-empty and up-connected as well as the region \bar{D}_1^{CH} is also non-empty and down-connected. We can define the free boundaries of American chooser options as

$$(3.8) \quad b_1^{CH}(t) = \sup\{x \in (0, \infty) \mid V^{CH}(t, x) \in \bar{D}_1^{CH}\},$$

$$(3.9) \quad b_2^{CH}(t) = \inf\{x \in (0, \infty) \mid V^{CH}(t, x) \in \bar{D}_2^{CH}\}.$$

The continuation region (3.1) and the optimal stopping regions in (3.6) and (3.7) can be expressed as

$$(3.10) \quad C^{CH} = \{(t, x) \in [0, T_1] \times (0, \infty) \mid b_1^{CH}(t) < x < b_2^{CH}(t)\},$$

$$(3.11) \quad \bar{D}_1^{CH} = \{(t, x) \in [0, T_1] \times (0, \infty) \mid x \leq b_1^{CH}(t)\},$$

$$(3.12) \quad \bar{D}_2^{CH} = \{(t, x) \in [0, T_1] \times (0, \infty) \mid x \geq b_2^{CH}(t)\},$$

where $b_1^{CH}(t) < g(t) < b_2^{CH}(t)$, for $t \in [0, T_1]$. For $t \leq T_1$, we have

$$(3.13) \quad C^{CH}(t) = \{x \in \mathbb{R}_+ \mid (t, x) \in C^{CH}\},$$

$$(3.14) \quad \bar{D}_1^{CH}(t) = \{x \in \mathbb{R}_+ \mid (t, x) \in \bar{D}_1^{CH}\},$$

$$(3.15) \quad \bar{D}_2^{CH}(t) = \{x \in \mathbb{R}_+ \mid (t, x) \in \bar{D}_2^{CH}\}.$$

These are the t-sections of continuation regions and optimal stopping regions, i.e. definition (3.13) is the continuation region for American chooser options for t given and fixed. Similarly, we can define $C^C(t)$ (continuation region for American call options at t), $\bar{D}^C(t)$, $C^P(t)$ (continuation region for American put options at t) and $\bar{D}^P(t)$.

In the following theorem, we want to show the region \bar{D}_1^{CH} is inside the optimal stopping region of American put options and region \bar{D}_2^{CH} is inside the optimal stopping region of American call options.

Theorem 2. *For any point $(t, x) \in \bar{D}_2^{CH}$, this implies $(t, x) \in \bar{D}^C$. The point $(t, x) \in \bar{D}_1^{CH}$ implies $(t, x) \in \bar{D}^P$.*

Proof. We first examine the case $L \leq K$: from Peskir (2005b) on American call and put options one can show that $C^C(t) \cap C^P(t)$ is non-empty for $t \in [0, T_1]$. By the result $C^C \cap C^P \subseteq C^{CH}$ from Theorem 1, it is clear that $\bar{D}_2^{CH} \subseteq \bar{D}^C$ and $\bar{D}_1^{CH} \subseteq \bar{D}^P$.

For the case $L > K$: assume that there exists a point $(t, x) \in \bar{D}_2^{CH}$ and $(t, x) \in C^C$. We therefore have $V^{CH}(t, x) = V^C(t, x)$ and it is optimal to hold the American call options rather than exercising it. Consider a portfolio consisting of a long position in American chooser options and a short position in American call options. The value of American chooser options is always larger than the value of American call options and we can exercise the chooser options to cover the value of American call options. If the holder of American call options exercises it at time $t' \in (t, T_1]$, the chooser option can be exercised to cover the short position of American call options. At the maturity T_1 , the value of chooser options is $V^{CH}(T_1, x) = \max(V^P(T_1, x), V^C(T_1, x))$. If $V^P(T_1, x) > V^C(T_1, x)$, we can exercise the chooser options to

get the American put options and the profit of the portfolio is $V^P(t, x) - V^C(t, x) > 0$ at T_1 . If $V^C(T_1, x) > V^P(T_1, x)$, we can exercise the chooser options to cover the short position of American call options. Since the probability $\tilde{P}(V^P(T_1, x) > V^C(T_1, x)) > 0$, the arbitrage opportunity of this portfolio exists. So this disproves the assumption at the beginning, we prove that the point $(t, x) \in \bar{D}_2^{CH}$ implies $(t, x) \in \bar{D}^C$ (i.e. $\bar{D}_2^{CH} \subseteq \bar{D}^C$). The proof of $\bar{D}_1^{CH} \subseteq \bar{D}^P$ can be shown in a similar way. \square

By the results of Theorems 1 and 2 we know the range of the upper and lower free boundaries for American chooser options

$$(3.16) \quad b_1^{CH}(t) < \min(g(t), b^P(t)),$$

$$(3.17) \quad b_2^{CH}(t) > \max(g(t), b^C(t)).$$

Since $b_1^{CH}(t) < b^P(t)$ and $b_2^{CH}(t) > b^C(t)$, if the chooser option exercises rationally to select the American put option before T_1 (the stock price entering the optimal stopping region \bar{D}_1^{CH}), it is optimal to exercise American put option immediately. If the chooser option exercises rationally to select the American call option before T_1 , it is optimal to exercise the American call option immediately.

4. American Chooser Option Properties

From the previous section, we gave the definition of value functions and the two free boundaries for the American chooser options. In this section we will prove that the value function is convex, binary continuous, and satisfies the smooth-fit property. The two free boundaries are monotonic, continuous and converge to a deterministic value at choosing maturity T_1 .

Theorem 3. *The value function $x \mapsto V^{CH}(t, x)$ is convex for $x \in (0, \infty)$.*

Proof. For any two points $x_1 < x_2$ from the domain $[0, \infty)$, and $\lambda \in [0, 1]$, we have

$$\begin{aligned} & V^{CH}(t, \lambda x_1 + (1 - \lambda)x_2) \\ &= \sup_{0 \leq \tau \leq T_1 - t} E_{t, \lambda x_1 + (1 - \lambda)x_2} [e^{-r\tau} G^{CH}(t + \tau, X_{t+\tau})], \\ &= \sup_{0 \leq \tau \leq T_1 - t} E[e^{-r\tau} [V^C(t + \tau, X_\tau^{\lambda x_1 + (1 - \lambda)x_2}) \vee V^P(t + \tau, X_\tau^{\lambda x_1 + (1 - \lambda)x_2})]], \\ &= \sup_{0 \leq \tau \leq T_1 - t} E[e^{-r\tau} [V^C(t + \tau, \lambda X_\tau^{x_1} + (1 - \lambda)X_\tau^{x_2}) \vee V^P(t + \tau, \lambda X_\tau^{x_1} + (1 - \lambda)X_\tau^{x_2})]], \end{aligned}$$

since $x \mapsto V^C(t, x)$ and $x \mapsto V^P(t, x)$ are both convex functions (see Peskir (2005b)),

$$\begin{aligned} & \leq \sup_{0 \leq \tau \leq T_1 - t} E[e^{-r\tau} [(\lambda V^C(t + \tau, X_\tau^{x_1}) + (1 - \lambda)V^C(t + \tau, X_\tau^{x_2})) \\ & \quad \vee (\lambda V^P(t + \tau, X_\tau^{x_1}) + (1 - \lambda)V^P(t + \tau, X_\tau^{x_2}))]], \\ & \leq \sup_{0 \leq \tau \leq T_1 - t} E[e^{-r\tau} [(\lambda V^C(t + \tau, X_\tau^{x_1}) \vee \lambda V^P(t + \tau, X_\tau^{x_1})) \\ & \quad + ((1 - \lambda)V^C(t + \tau, X_\tau^{x_2}) \vee (1 - \lambda)V^P(t + \tau, X_\tau^{x_2}))]], \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{0 \leq \tau \leq T_1 - t} E[e^{-r\tau} (\lambda V^C(t + \tau, X_\tau^{x_1}) \vee \lambda V^P(t + \tau, X_\tau^{x_1}))] \\
&\quad + \sup_{0 \leq \tau \leq T_1 - t} E[e^{-r\tau} ((1 - \lambda) V^C(t + \tau, X_\tau^{x_2}) \vee (1 - \lambda) V^P(t + \tau, X_\tau^{x_2}))], \\
(4.1) \quad &= \lambda V^{CH}(t, x_1) + (1 - \lambda) V^{CH}(t, x_2).
\end{aligned}$$

So $x \mapsto V^{CH}(t, x)$ is a convex function, where X_t^x denotes the value of a stochastic process X at time t , starting with $X_0 = x$. \square

We now prove the free boundary property and the continuity of the binary function.

Theorem 4. *The free boundaries of American chooser options satisfy*

- (i) *function $b_1^{CH}(t)$ is an increasing function for $t \in [0, T_1]$ and $b_1^{CH}(T_1 -) = \min(b_1^P(T_1), g(T_1))$;*
- (ii) *function $b_2^{CH}(t)$ is a decreasing function for $t \in [0, T_1]$ and $b_2^{CH}(T_1 -) = \max(b_1^C(T_1), g(T_1))$.*

Secondly, the value function $(t, x) \mapsto V^{CH}(t, x)$ is a continuous function in $[0, T_1] \times (0, \infty)$.

Proof. Firstly, the free boundaries of American chooser options are proved by Proposition 3 in Detemple and Emmerling (2009) for $L \leq K$. For $L > K$, the proof is given in Proposition 8 in Detemple and Emmerling (2009). Secondly, the continuity of the binary function is proven as follows. The continuity of the binary function is equivalent to

$$(4.2) \quad x \mapsto V^{ST}(t, x) \text{ is continuous } \in [0, \infty),$$

$$(4.3) \quad t \mapsto V^{ST}(t, x) \text{ is uniformly continuous in } [0, T_1) \text{ for } x \text{ given and fixed.}$$

Since $x \mapsto V^{CH}(t, x)$ is a convex function, the value function is continuous in x . It remains to prove $t \mapsto V^{CH}(t, x)$ is uniformly continuous. Given arbitrary $0 \leq t_1 < t_2 \leq T$ and $x \in (0, \infty)$, and let τ_1 be the optimal stopping time for $V^{CH}(t_1, x)$. Set $\tau_2 = \tau_1 \wedge (T - t_1)$, it is clear that $\tau_2 \leq \tau_1$ and $\tau_2 \leq T - t_1$. If $\tau_1 < T_1$, the proof is the same as Theorem 6 in Qiu (2014). We assume $\tau_1 = T_1$, we have

$$\begin{aligned}
(4.4) \quad &V^{CH}(t_1, x) - V^{CH}(t_2, x) \\
&\leq E \left[e^{-r(T_1 - t_2)} \left[\max(V^C(T_1, X_{T_1 - t_1}^x), V^P(T_1, X_{T_1 - t_1}^x)) \right. \right. \\
&\quad \left. \left. - \max(V^C(T_1, X_{T_1 - t_2}^x), V^P(T_1, X_{T_1 - t_2}^x)) \right] \right], \\
&\leq E \left[e^{-r(T_1 - t_2)} \max \left(V^C(T_1, X_{T_1 - t_1}^x) - V^C(T_1, X_{T_1 - t_2}^x) \right. \right. \\
&\quad \left. \left. V^P(T_1, X_{T_1 - t_2}^x) - V^P(T_1, X_{T_1 - t_2}^x) \right) \right], \\
&\leq e^{-r(T_1 - t_2)} E \left[\sup_{0 \leq t \leq t_2 - t_1} \max \left(V^P(T_1, X_{T_1 - t_1}^x) - V^P(T_1, X_{T_1 - t_1 - t}^x), \right. \right. \\
&\quad \left. \left. V^C(T_1, X_{T_1 - t_1}^x) - V^C(T_1, X_{T_1 - t_1 - t}^x) \right) \right] =: e^{-r(T_1 - t_2)} L(t_2 - t_1).
\end{aligned}$$

Since $V^C(t, x)$ and $V^P(t, x)$ are continuous functions in $[0, T_1] \times (0, \infty)$, the function $L(t_2 - t_1) \rightarrow 0$ as $t_2 - t_1 \rightarrow 0$.

$$(4.5) \quad 0 \leq V^{CH}(t_1, x) - V^{CH}(t_2, x) \leq e^{-r(T_1 - t_2)} L(t_2 - t_1) \leq L(t_2 - t_1).$$

The statement equation 4.3 holds, and the proof is completed. \square

We now prove the smooth fit property of American chooser options.

Theorem 5. *The value function satisfies the smooth fit property, that is*

$$(4.6) \quad \left. \frac{\partial V^{CH}(t, x)}{\partial x} \right|_{x=b_1^{CH}(t)} = -1,$$

$$(4.7) \quad \left. \frac{\partial V^{CH}(t, x)}{\partial x} \right|_{x=b_2^{CH}(t)} = 1,$$

with $0 \leq t < T_1$. Also the boundaries of American chooser options $t \mapsto b_1^{CH}(t)$ and $t \mapsto b_2^{CH}(t)$ are continuous for $t \in [0, T_1]$.

Proof. The proofs for (4.7), and (4.6) can be illustrated in the same way. Let $x = b^{CH}(t)$ and $(t, x) \in [0, T_1] \times (0, \infty)$.

1. Since $x = b_2^{CH}(t) \geq b^C(t) > K$, there exists $\varepsilon > 0$ such that $x - \varepsilon > K$. We have the inequality

$$(4.8) \quad \frac{V^{CH}(t, x) - V^{CH}(t, x - \varepsilon)}{\varepsilon} \leq \frac{x - K - (x - \varepsilon - K)}{\varepsilon} = 1,$$

and ε approaching to 0 makes

$$(4.9) \quad \frac{\partial^- V^{CH}(t, x)}{\partial x} \leq 1.$$

2. With fixed $\varepsilon > 0$, assume τ_ε is the optimal stopping time to $V^{CH}(t, x - \varepsilon)$. Let us set $\gamma = r - \delta - \sigma^2/2$ and by the proof of Theorem 6 in Qiu (2014), we have $\tau_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. By the definition of τ_ε , we have

$$\begin{aligned} V^{CH}(t, x) - V^{CH}(t, x - \varepsilon) &\geq E[e^{-r\tau_\varepsilon} G^{CH}(t + \tau_\varepsilon, X_{\tau_\varepsilon}^x)] - E[e^{-r\tau_\varepsilon} G^{CH}(t + \tau_\varepsilon, X_{\tau_\varepsilon}^{x-\varepsilon})], \\ &= E[e^{-r\tau_\varepsilon} (G^{CH}(t + \tau_\varepsilon, X_{\tau_\varepsilon}^x) - G^{CH}(t + \tau_\varepsilon, X_{\tau_\varepsilon}^{x-\varepsilon}))]. \end{aligned}$$

Since $X_{\tau_\varepsilon}^{x+\varepsilon} > X_{\tau_\varepsilon}^x$, $\bar{D}_2^{CH} \subseteq \bar{D}^C$ and $\bar{D}_1^{CH} \subseteq \bar{D}^P$ from Theorem 2, the right hand side from the above equation can be separated into

$$\begin{aligned} &\geq E[e^{-r\tau_\varepsilon} (L - X_{\tau_\varepsilon}^x - L + X_{\tau_\varepsilon}^{x-\varepsilon}) I(X_{\tau_\varepsilon}^{x-\varepsilon} < g(t + \tau_\varepsilon), \tau_\varepsilon \neq T_1 - t)] \\ &\quad + E[e^{-r\tau_\varepsilon} (X_{\tau_\varepsilon}^x - K - X_{\tau_\varepsilon}^{x-\varepsilon} + K) I(X_{\tau_\varepsilon}^{x-\varepsilon} \geq g(t + \tau_\varepsilon), \tau_\varepsilon \neq T_1 - t)] \\ &\quad + E[e^{-r\tau_\varepsilon} (V^C(t + \tau_\varepsilon, X_{\tau_\varepsilon}^x) - V^C(t + \tau_\varepsilon, X_{\tau_\varepsilon}^{x-\varepsilon})) I(X_{\tau_\varepsilon}^{x-\varepsilon} < g(t + \tau_\varepsilon), \tau_\varepsilon = T_1 - t)] \\ &\quad + E[e^{-r\tau_\varepsilon} (V^P(t + \tau_\varepsilon, X_{\tau_\varepsilon}^x) - V^P(t + \tau_\varepsilon, X_{\tau_\varepsilon}^{x-\varepsilon})) I(X_{\tau_\varepsilon}^{x-\varepsilon} \geq g(t + \tau_\varepsilon), \tau_\varepsilon = T_1 - t)] \end{aligned}$$

$$\begin{aligned}
&\geq -\varepsilon E[e^{-r\tau_\varepsilon} \exp(\sigma B_{\tau_\varepsilon} + \gamma\tau_\varepsilon) I(X_{\tau_\varepsilon}^{x-\varepsilon} < g(t + \tau_\varepsilon), \tau_\varepsilon \neq T_1 - t)] \\
&\quad + \varepsilon E[e^{-r\tau_\varepsilon} \exp(\sigma B_{\tau_\varepsilon} + \gamma\tau_\varepsilon) I(X_{\tau_\varepsilon}^{x-\varepsilon} \geq g(t + \tau_\varepsilon), \tau_\varepsilon \neq T_1 - t)] \\
&\quad + \varepsilon E\left[e^{\sigma B_{T_1-t} + (\gamma-r)(T_1-t)} \frac{V^P(T_1, X_{T_1-t}^{x-\varepsilon}) - V^P(T_1, X_{T_1-t}^x)}{-\varepsilon \exp(\sigma B_{T_1-t} + \gamma(T_1 - t))} I(X_{T_1-t}^{x-\varepsilon} \geq g(T_1), \tau_\varepsilon = T_1 - t)\right],
\end{aligned}
\tag{4.10}$$

where $I(\cdot)$ is the indicator function. If we divide ε on both sides of inequality (4.10) and taking $\varepsilon \rightarrow 0$, as we know $x = b_2^{CH}(t) > g(t)$ and

$$\frac{V^P(T_1, X_{T_1-t}^{x-\varepsilon}) - V^P(T_1, X_{T_1-t}^x)}{-\varepsilon \exp(\sigma B_{T_1-t} + \gamma(T_1 - t))} \rightarrow V_x^P(T_1, x) = -1,$$

the first term and the third term from the right hand side of inequality (4.10) go to zero. So we have

$$(4.11) \quad \frac{\partial V^{CH}(t, x)}{\partial x} \Big|_{x=b_2^{CH}(t)} = \lim_{\varepsilon \rightarrow 0^+} \frac{V^{CH}(t, b_2^{CH}(t)) - V^{CH}(t, b_2^{CH}(t) - \varepsilon)}{\varepsilon} \geq 1$$

From inequality (4.9) and (4.11), the proof of (4.7) is completed.

The proof that the boundaries of American chooser options $t \mapsto b_1^{CH}(t)$ and $t \mapsto b_2^{CH}(t)$ are continuous for $t \in [0, T_1)$ can be given as follows. Since $\bar{D}_2^{CH} \subseteq \bar{D}^C$ and $\bar{D}_1^{CH} \subseteq \bar{D}^P$ from Theorem 2, the payoff of American chooser options in the stopping region is the same as the payoff of American strangle options before T_1 . So the proof of continuity for the free boundaries of American chooser options is completely the same as the proof in Theorem 9 of Qiu (2014). \square

5. The EEP Representation Of American Chooser Options

The American type options enjoy the facility of early exercise before the maturity T_1 , therefore the American option holder pays extra for this facility compared to the equivalent European option. The extra payment is called the early exercise premium (EEP). In this section we will use the local time-space formula (see Peskir (2005a)) to obtain the EEP representation for the value of American chooser options, i.e. the value of European chooser options plus the early exercise premium. With standard arguments (see Peskir and Shiryaev (2005)) based on the strong Markov property, we derive the free-boundary problem for unknown $V^{CH}(t, x)$ and two unknown boundaries $b_1^{CH}(t)$ and $b_2^{CH}(t)$ for $(t, x) \in [0, T_1) \times (0, \infty)$:

$$\begin{aligned}
(5.1) \quad &V_t^{CH} + \mathbb{L}_X V^{CH} = rV^{CH} \quad \text{in } C^{CH}, \\
(5.2) \quad &V_x^{CH}(t, x) = -1 \quad \text{for } x = b_1^{CH}(t), t < T_1, \\
(5.3) \quad &V_x^{CH}(t, x) = 1 \quad \text{for } x = b_2^{CH}(t), t < T_1, \\
(5.4) \quad &V^{CH}(t, x) = G^{CH}(t, x) = L - x \quad \text{for } x \leq b_1^{CH}(t), t < T_1, \\
(5.5) \quad &V^{CH}(t, x) = G^{CH}(t, x) = x - K \quad \text{for } x \geq b_2^{CH}(t), t < T_1, \\
(5.6) \quad &V^{CH}(T_1, x) = G^{CH}(T_1, x) = V^P(T_1, x) \quad \text{for } x = b_1^{CH}(T_1), \\
(5.7) \quad &V^{CH}(T_1, x) = G^{CH}(T_1, x) = V^C(T_1, x) \quad \text{for } x = b_2^{CH}(T_1), \\
(5.8) \quad &V^{CH}(t, x) > G^{CH}(t, x) \quad \text{in } C^{CH}.
\end{aligned}$$

If we consider Theorems 3 to 5, the value function V^{CH} and two free boundaries have the following properties

- (5.9) $V^{CH}(t, x)$ is continuous function in $[0, T_1] \times (0, \infty)$,
- (5.10) V^{CH} is $C^{1,2}$ in C^{CH} , \bar{D}_1^{CH} and \bar{D}_2^{CH} except for (T_1, x) , $x \in (0, \infty)$,
- (5.11) $x \mapsto V^{CH}(t, x)$ is convex function with $V_x^{CH}(t, x) \in [-1, 1]$,
- (5.12) $t \mapsto V^{CH}(t, x)$ is decreasing function with $V^{CH}(T_1, x) = G^{CH}(T_1, x)$,
 b_1^{CH} is increasing function in $[0, T_1]$, and continuous with $0 < b_1^{CH}(t) < L$,
- (5.13) in $[0, T_1)$, and $b_1^{CH}(T_1-) = \min(g(T_1), b^P(T_1))$,
 b_2^{CH} is decreasing function in $[0, T_1]$, and continuous with $K < b_2^{CH}(t) < \infty$,
- (5.14) in $[0, T_1)$, and $b_2^{CH}(T_1-) = \max(g(T_1), b^C(T_1))$.

By (5.2), (5.3), (5.10) and (5.11), we can apply the change of variable formula with local time on curves to $e^{-rs}V^{CH}(t+s, X_{t+s})$ and take the $P_{t,x}$ expectation from both sides. By the Optional Sampling Theorem, the martingale term will disappear. Finally, using equations (5.1), (5.4), (5.5) and taking $s = T_1 - t$, we obtain the EEP representation of American chooser options

$$\begin{aligned}
 V^{CH}(t, x) &= E_{t,x} e^{-r(T_1-t)} V^{CH}(T_1, X_{T_1}) \\
 &\quad - E_{t,x} \int_0^{T_1-t} (-rL + \delta X_{t+s}) I(X_{t+s} \leq b_1^{CH}(t+s)) ds \\
 &\quad - E_{t,x} \int_0^{T_1-t} (rK - \delta X_{t+s}) I(X_{t+s} \geq b_2^{CH}(t+s)) ds.
 \end{aligned}
 \tag{5.15}$$

By the property of Brownian motion, and after straight forward computation, we obtain the computational form of the EEP representation

$$\begin{aligned}
 V^{CH}(t, x) &= E_{t,x} e^{-r(T_1-t)} V^{CH}(T_1, X_{T_1}) \\
 &\quad + \int_0^{T_1-t} e^{-rs} r L \Phi(d_1(s, b_1^{CH}(t+s), x)) - e^{-\delta s} \delta x \Phi(d_1(s, b_1^{CH}(t+s), x) - \sigma\sqrt{s}) ds \\
 &\quad + \int_0^{T_1-t} e^{-\delta s} \delta x \Phi(d_2(s, b_2^{CH}(t+s), x) + \sigma\sqrt{s}) - e^{-rs} r K \Phi(d_2(s, b_2^{CH}(t+s), x)) ds,
 \end{aligned}
 \tag{5.16}$$

where

$$\begin{aligned}
 d_1(s, b_1^{CH}(t+s), x) &= \frac{1}{\sigma\sqrt{s}} \left(\ln \frac{b_1^{CH}(t+s)}{x} - \left(r - \delta - \frac{\sigma^2}{2} \right) s \right), \\
 d_2(s, b_2^{CH}(t+s), x) &= -\frac{1}{\sigma\sqrt{s}} \left(\ln \frac{b_2^{CH}(t+s)}{x} - \left(r - \delta - \frac{\sigma^2}{2} \right) s \right),
 \end{aligned}$$

and $\Phi(\cdot)$ is the standard normal distribution function, the first term in (5.16) is the value of European chooser options. For $0 \leq t < T_1$, the free boundaries can be given by the implicit functions

$$L - b_1^{CH}(t) = E_{t, b_1^{CH}(t)} e^{-r(T_1-t)} V^{CH}(T_1, X_{T_1})
 \tag{5.17}$$

$$\begin{aligned}
& + \int_0^{T_1-t} e^{-rs} r L \Phi(d_1(s, b_1^{CH}(t+s), b_1^{CH}(t))) ds \\
& - \int_0^{T_1-t} e^{-\delta s} \delta b_1^{CH}(t) \Phi(d_1(s, b_1^{CH}(t+s), b_1^{CH}(t)) + \sigma \sqrt{s}) ds \\
& + \int_0^{T_1-t} e^{-\delta s} \delta b_1^{CH}(t) \Phi(d_2(s, b_2^{CH}(t+s), b_1^{CH}(t)) - \sigma \sqrt{s}) ds \\
& - \int_0^{T_1-t} e^{-rs} r K \Phi(d_2(s, b_2^{CH}(t+s), b_1^{CH}(t))) ds, \\
(5.18) \quad b_2^{CH}(t) - K & = E_{t, b_2^{CH}(t)} e^{-r(T_1-t)} V^{CH}(T_1, X_{T_1}) \\
& + \int_0^{T_1-t} e^{-rs} r L \Phi(d_1(s, b_1^{CH}(t+s), b_2^{CH}(t))) ds \\
& - \int_0^{T_1-t} e^{-\delta s} \delta b_2^{CH}(t) \Phi(d_1(s, b_1^{CH}(t+s), b_2^{CH}(t)) + \sigma \sqrt{s}) ds \\
& + \int_0^{T_1-t} e^{-\delta s} \delta b_2^{CH}(t) \Phi(d_2(s, b_2^{CH}(t+s), b_2^{CH}(t)) - \sigma \sqrt{s}) ds \\
& - \int_0^{T_1-t} e^{-rs} r K \Phi(d_2(s, b_2^{CH}(t+s), b_2^{CH}(t))) ds.
\end{aligned}$$

From the free-boundary problem, we form the system by (5.17) and (5.18). The free boundaries defined in (3.9) and (3.10) are the solution pair of (5.17) and (5.18).

For the stopping region for American chooser options \bar{D}_1^{CH} and \bar{D}_2^{CH} given in (3.7) and (3.8), the payoff function is the same as American strangle before choosing maturity T_1 . After applying Peskir and Shiryaev (2005), we can use a simplified way to prove the uniqueness of the free boundary. Before starting the proof, we need to assert two theorems.

Theorem 6. *If $X = (X_t)_{t \geq 0}$ is a Markov process and we set $F(t, x) = E_{t,x} G(T, X_T)$ for a (bounded) measurable function G with $P_{t,x}(X_t = x) = 1$, then $F(t+s, X_{t+s})$ is a martingale under $P_{t,x}$ for $0 \leq s \leq T-t$.*

Proof. From the definition of $F(t, x)$, we have $F(t+s, X_{t+s}) = E_{t+s, X_{t+s}} G(T, X_T)$. For $\forall s' < s$

$$\begin{aligned}
& E_{t,x}[F(t+s, X_{t+s}) | \mathcal{F}_{t+s'}] \\
& = E_{t,x}[E_{t+s, X_{t+s}} G(T, X_T) | \mathcal{F}_{t+s'}], \\
& = E_{t,x}[E_{t,x} G(T, X_T) | \mathcal{F}_{t+s} | \mathcal{F}_{t+s'}] \quad (\text{Markov property of } X), \\
& = E_{t,x}[G(T, X_T) | \mathcal{F}_{t+s'}] \quad (\text{Property of conditional expectation}), \\
& = E_{t+s', X_{t+s'}}[G(T, X_T)] \quad (\text{Markov property of } X), \\
& = F(t+s', X_{t+s'}).
\end{aligned}$$

□

Theorem 7. *If $X = (X_t)_{t \geq 0}$ is a Markov process and we define a function that $F(t, x) = E_{t,x} \int_0^{T-t} H(X_{t+u}) du$ for a (bounded) measurable function H with $P_{t,x}(X_t = x) = 1$, then $F(t+s, X_{t+s}) + \int_0^s H(X_{t+u}) du$ is a martingale under $P_{t,x}(X_t = x) = 1$ for $0 \leq s \leq T-t$.*

Proof. For $\forall s' < s$

$$\begin{aligned}
& E_{t,x} \left[F(t+s, X_{t+s}) + \int_0^s H(t+u, X_{t+u}) du | \mathcal{F}_{t+s'} \right] \\
&= E_{t,x} \left[E_{t+s, X_{t+s}} \int_0^{T-t-s} H(t+s+u, X_{t+s+u}) du + \int_0^s H(t+u, X_{t+u}) du | \mathcal{F}_{t+s'} \right], \\
&= E_{t,x} \left[E_{t,x} \left[\int_s^{T-t} H(t+u, X_{t+u}) du | \mathcal{F}_{t+s} \right] + \int_0^s H(t+u, X_{t+u}) du | \mathcal{F}_{t+s'} \right], \\
&= E_{t,x} \left[\int_s^{T-t} H(t+u, X_{t+u}) du | \mathcal{F}_{t+s'} \right] + E_{t,x} \left[\int_0^{s'} H(t+u, X_{t+u}) du + \int_{s'}^s H(t+u, X_{t+u}) du | \mathcal{F}_{t+s'} \right], \\
&= E_{t,x} \left[\int_{s'}^{T-t} H(t+u, X_{t+u}) du | \mathcal{F}_{t+s'} \right] + \int_0^{s'} H(t+u, X_{t+u}) du, \\
&= E_{t+s', X_{t+s'}} \left[\int_{s'}^{T-t} H(t+u, X_{t+u}) du \right] + \int_0^{s'} H(t+u, X_{t+u}) du, \\
&= E_{t+s', X_{t+s'}} \left[\int_0^{T-t-s'} H(t+s'+u, X_{t+s'+u}) du \right] + \int_0^{s'} H(t+u, X_{t+u}) du, \\
&= F(t+s', X_{t+s'}) + \int_0^{s'} H(t+u, X_{t+u}) du.
\end{aligned}$$

□

With the Theorems 6 and 7, we can now prove Theorem 8.

Theorem 8. *The optimal stopping boundaries (free boundaries) of American chooser options (2.8) can be characterized as the unique solution pair of the system including (5.17) and (5.18). The solution of the lower boundary is in the class of continuous increasing function $c_1 : [0, T_1) \rightarrow \mathbb{R}_+$ satisfying $0 \leq c_1(t) \leq \min(b^P(t), g(t))$ for $t \in [0, T_1)$ and $c_1(T_1-) = \min(b^P(T_1), g(T_1))$. The solution of the upper boundary is in the class of continuous decreasing function $c_2 : [0, T_1) \rightarrow \mathbb{R}$ satisfying $\max(b^C(t), g(t)) < c_2(t) < \infty$ for $t \in [0, T_1)$ and $c_2(T_1-) = \max(b^C(T_1), g(T_1))$.*

Proof. The definition of free boundaries b_1^{CH} and b_2^{CH} , and the EEP representation of chooser options in (5.15) shows that b_1^{CH} and b_2^{CH} are the solution of (5.17) and (5.18). In the following, we will use five steps to prove this solution pair is unique (i.e. prove Theorem 8).

1. Let $c_1(t)$ and $c_2(t)$ be another set of solutions of the system including (5.17) and (5.18), and this solution pair satisfies the properties mentioned in Theorem 8. Putting c_1 and c_2 into equation (5.15), we obtain

$$\begin{aligned}
(5.19) \quad U^c(t, x) &= E_{t,x} e^{-r(T_1-t)} V^{CH}(T_1, X_{T_1}) \\
&- E_{t,x} \int_0^{T_1-t} (-rL + \delta X_{t+s}) I(X_{t+s} \leq c_1(t+s)) ds \\
&- E_{t,x} \int_0^{T_1-t} (rK - \delta X_{t+s}) I(X_{t+s} \geq c_2(t+s)) ds.
\end{aligned}$$

Applying the Theorems 6 and 7, we can show that

$$(5.20) \quad \begin{aligned} & e^{-rs}U^c(t+s, X_{t+s}) - \int_0^s e^{-ru}(-rL + \delta X_{t+u})I(X_{t+u} \leq c_1(t+u))du \\ & - \int_0^s e^{-ru}(rK - \delta X_{t+u})I(X_{t+u} \geq c_2(t+u))du \end{aligned}$$

is a continuous martingale in $\mathcal{P}_{t,x}$. We will give a brief illustration on (5.20) is a martingale. We set

$$(5.21) \quad H(t, x) = (-rL + \delta x)I(x \leq c_1(t)) + (rK - \delta x)I(x \geq c_2(t))$$

The stochastic process (5.20) can be written as

$$(5.22) \quad \begin{aligned} & e^{-r(T_1-t)}E_{t+s, X_{t+s}}G(T_1, X_{T_1}) - E_{t+s, X_{t+s}} \int_0^{T_1-t-s} e^{-r(u+s)}H(t+s+u, X_{t+s+u})du \\ & - \int_0^s e^{-ru}H(t+u, X_{t+u})du. \end{aligned}$$

Theorem 6 shows that the first term in equation (5.22) is a martingale, and Theorem 7 shows that the second term plus the last term in equation (5.22) is another martingale as well. Therefore we conclude that equation (5.20) is a martingale. We know c_1 and c_2 are the solutions of (5.17) and (5.18), meanwhile $0 \leq c_1(t) \leq \min(b^P(t), g(t))$ and $\max(b^P(t), g(t)) \leq c_2(t) \leq \infty$. So it is obvious to obtain

$$(5.23) \quad G^{CH}(t, c_1(t)) = L - c_1(t) = U^c(t, c_1(t)),$$

$$(5.24) \quad G^{CH}(t, c_2(t)) = c_2(t) - K = U^c(t, c_2(t)).$$

We set the stopping time

$$(5.25) \quad \sigma_{c_1} = \inf\{s \in [0, T_1 - t] \mid X_s^x \geq c_1(t+s)\}.$$

Applying local time-space formula to $e^{-rs}G^{CH}(t+s, X_{t+s})$, we obtain

$$(5.26) \quad \begin{aligned} e^{-rs}G^{CH}(t+s, X_{t+s}) &= G^{CH}(t, x) \\ &+ \int_0^s e^{-ru}(V_t^P + \mathbb{L}_X V^P - rV^P)(t+u, X_{t+u})I(X_{t+u} < g(t+u))du \\ &+ \int_0^s e^{-ru}(V_t^C + \mathbb{L}_X V^C - rV^C)(t+u, X_{t+u})I(X_{t+u} > g(t+u))du \\ &+ M_s + \int_0^s e^{-ru}d\ell_u^{X^*}(X). \end{aligned}$$

For $x \leq c_1(t)$, since (5.20) is a martingale, using the Optional Sampling Theorem, we obtain

$$(5.27) \quad U^c(t, x) = E_{t,x}e^{-r\sigma_{c_1}}U^c(t+\sigma_{c_1}, X_{t+\sigma_{c_1}}) - E_{t,x} \int_0^{\sigma_{c_1}} e^{-ru}(-rL + \delta X_{t+u})du.$$

Using σ_{c_1} to replace s in equation (5.26), since $c_1(t) \leq \min(g(t), b^P(t))$, we have that

$$(V_t^P + \mathbb{L}_X V^P - rV^P)(t+u, X_{t+u}) = -rL + \delta X_{t+u},$$

for $u \in (0, \sigma_{c_1})$ such that equation (5.26) can be written as

$$(5.28) \quad E_{t,x} e^{-r\sigma_{c_1}} U^c(t + \sigma_{c_1}, X_{t+\sigma_{c_1}}) = G^{CH}(t, x) + E_{t,x} \int_0^{\sigma_{c_1}} e^{-ru} (-rL + \delta X_{t+u}) du.$$

If we substitute equation (5.28) into equation (5.27), we obtain

$$(5.29) \quad U^c(t, x) = G^{CH}(t, x) = L - x.$$

Similarly, we can prove

$$(5.30) \quad U^c(t, x) = G^{CH}(t, x) = x - K,$$

for $x \geq c_2(t)$, $t \in [0, T_1]$.

2. In this part, we want to show $U^c(t, x) \leq V^{CH}(t, x)$ for $(t, x) \in [0, T_1] \times (0, \infty)$. By the martingale in equation (5.20), we can apply the Optional Sampling Theorem and obtain

$$(5.31) \quad \begin{aligned} U^c(t, x) = & E_{t,x} U^c(t + \sigma'_c, X_{t+\sigma'_c}) - E_{t,x} \int_0^{\sigma'_c} e^{-ru} (-rL + \delta X_{t+u}) I(X_{t+u} \leq c_1(t+u)) du \\ & - E_{t,x} \int_0^{\sigma'_c} e^{-ru} (rK - \delta X_{t+u}) I(X_{t+u} \geq c_2(t+u)) du, \end{aligned}$$

where σ'_c is defined as

$$(5.32) \quad \sigma'_c = \inf\{s \in [0, T_1 - t] \mid X_{t+s} \leq c_1(t+s) \text{ or } X_{t+s} \geq c_2(t+s)\}.$$

If $x \leq c_1(t)$ or $x \geq c_2(t)$, then $\sigma'_c = 0$. So

$$(5.33) \quad U^c(t, x) = G^{CH}(t, x) \leq V^{CH}(t, x).$$

If $c_1(t) < x < c_2(t)$, equation (5.31) will change to

$$(5.34) \quad U^c(t, x) = E_{t,x} U^c(t + \sigma'_c, X_{t+\sigma'_c}) = E_{t,x} G^{CH}(t + \sigma'_c, X_{t+\sigma'_c}) \leq V^{CH}(t, x).$$

If we combine equations (5.33) and (5.34), we prove that

$$(5.35) \quad U^c(t, x) \leq V^{CH}(t, x).$$

3. This part will prove $b_1^{CH}(t) \leq c_1(t)$ for $t \in [0, T_1]$. Suppose a stopping time

$$(5.36) \quad \sigma_{b_1} = \inf\{s \in [0, T_1 - t] \mid X_s^x \geq b_1^{CH}(t+s)\} \wedge T_1 - t,$$

where $x \leq c_1(t) \wedge b_1^{CH}(t)$. Applying local time-space formula to $e^{-rs} V^{CH}(t+s, X_{t+s})$ with $X_t = x$, we obtain

$$V^{CH}(t, x) = e^{-rs} V^{CH}(t+s, X_{t+s}) - \int_0^s e^{-ru} (-rL + \delta X_{t+u}) I(X_{t+u} < b_1^{CH}(t+u)) du$$

$$(5.37) \quad - \int_0^s e^{-ru} (rK - \delta X_{t+u}) I(X_{t+u} > b_2^{CH}(t+u)) du + M_s.$$

Substituting s by σ_{b_1} into equation (5.37) and taking expectation in $P_{t,x}$ from both sides, we obtain the equation

$$(5.38) \quad E_{t,x} e^{-r\sigma_{b_1}} V^{CH}(t + \sigma_{b_1}, X_{t+\sigma_{b_1}}) = V^{CH}(t, x) + E_{t,x} \int_0^{\sigma_{b_1}} e^{-ru} (-rL + \delta X_{t+u}) du.$$

Since equation (5.20) is a martingale, by the Optional Sampling Theorem, we obtain

$$(5.39) \quad E_{t,x} e^{-r\sigma_{b_1}} U^c(t + \sigma_{b_1}, X_{t+\sigma_{b_1}}) = U^c(t, x) + E_{t,x} \int_0^{\sigma_{b_1}} e^{-ru} (-rL + \delta X_{t+u}) I(X_{t+u} \leq c_1(t+u)) du.$$

Since $x \leq c_1(t) \wedge b_1^{CH}(t) \leq c_1(t)$, by part 1, it is obvious that $U^c(t, x) = G^{CH}(t, x) = V^{CH}(t, x)$. As we know $U^c(t, x) \leq V^{CH}(t, x)$ for $(t, x) \in [0, T_1] \times (0, \infty)$, so

$$(5.40) \quad U^c(t + \sigma_{b_1}, X_{t+\sigma_{b_1}}) \leq V^{CH}(t + \sigma_{b_1}, X_{t+\sigma_{b_1}}).$$

After comparing equations (5.38) and (5.39), we have

$$(5.41) \quad E_{t,x} \int_0^{\sigma_{b_1}} e^{-ru} (rL - \delta X_{t+u}) I(X_{t+u} \leq c_1(t+u)) du \geq E_{t,x} \int_0^{\sigma_{b_1}} e^{-ru} (rL - \delta X_{t+u}) du.$$

Since $rL - \delta X_{t+u} \geq 0$ for $X_{t+u} \leq b_1^{CH}(t+u) \leq b_1^{CH}(T_1-)$, so $c_1(t+u) \geq b_1^{CH}(t+u)$ for $u \in [0, \sigma_{b_1}]$. Let $u = 0$, we obtain $c_1(t) \geq b_1^{CH}(t)$ for $t \in [0, T_1]$.

4. In this part, we will prove $b_2^{CH} \geq c_2(t)$ for $t \in [0, T_1]$. Take $x \geq c_2(t) \vee b_2^{CH}(t)$ for $t \in [0, T_1]$. Let

$$(5.42) \quad \sigma_{b_2} = \inf\{s \in [0, T_1 - t] \mid X_s^x \leq b_2(t+s)\} \wedge T_1 - t,$$

be a stopping time. Substituting σ_{b_2} into s for the equation (5.37), we obtain

$$(5.43) \quad \begin{aligned} V^{CH}(t, x) &= e^{-r\sigma_{b_2}} V^{CH}(t + \sigma_{b_2}, X_{t+\sigma_{b_2}}) \\ &\quad - \int_0^{\sigma_{b_2}} e^{-rs} (rK - \delta X_{t+s}) I(X_{t+s} \geq b_2^{CH}(t+s)) ds + M_{\sigma_{b_2}}. \end{aligned}$$

Taking the expectation in $P_{t,x}$ for (5.43), we obtain the equation

$$(5.44) \quad \begin{aligned} E_{t,x} e^{-r\sigma_{b_2}} V^{CH}(t + \sigma_{b_2}, X_{t+\sigma_{b_2}}) &= V^{CH}(t, x) \\ &\quad + E_{t,x} \int_0^{\sigma_{b_2}} (rK - \delta X_{t+s}) I(X_{t+s} \geq b_2^{CH}(t+s)) ds. \end{aligned}$$

Since equation (5.20) is a martingale, we use the Optional Sampling Theorem to obtain

$$(5.45) \quad \begin{aligned} U^c(t, x) &= E_{t,x} e^{-r\sigma_{b_2}} U^c(t + \sigma_{b_2}, X_{t+\sigma_{b_2}}) \\ &\quad - E_{t,x} \int_0^{\sigma_{b_2}} e^{-rs} (rK - \delta X_{t+s}) I(X_{t+s} \geq c_2(t+s)) ds. \end{aligned}$$

Equality (5.45) can be written as

$$(5.46) \quad \begin{aligned} E_{t,x} e^{-r\sigma_{b_2}} U^c(t + \sigma_{b_2}, X_{\sigma_{b_2}}) &= U^c(t, x) \\ &+ E_{t,x} \int_0^{\sigma_{b_2}} e^{-rs} (rK - \delta X_{t+s}) I(X_{t+s} \geq c_2(t+s)) ds. \end{aligned}$$

Comparing equations (5.44) and (5.45), we easily get that

$$(5.47) \quad E_{t,x} \int_0^{\sigma_{b_2}} e^{-rs} (\delta X_{t+s} - rK) ds \leq E_{t,x} \int_0^{\sigma_{b_2}} (\delta X_{t+s} - rK) I(X_{t+s} \geq c_2(t+s)) ds.$$

Since $\delta X_{t+s} - rK \geq 0$ for $s \in [0, \sigma_{b_2}]$, the inequality (5.47) indicates that $c_2(t+s) \leq b_2^{CH}(t+s)$ for $s \in [0, \sigma_{b_2}]$. Let s be zero, we have $c_2(t) \leq b_2^{CH}(t)$ for $t \in [0, T_1]$.

5. In the final part, we will prove that $c_1(t) = b_1^{CH}(t)$ and $c_2(t) = b_2^{CH}(t)$ for $t \in [0, T_1]$. We assume that there exist $t \in [0, T_1]$, such that $c_1(t) > b_1^{CH}(t)$. So there exists $x \in (b_1^{CH}(t), c_1(t))$. We define a stopping time

$$(5.48) \quad \tau_b = \inf\{0 \leq t \leq T_1 - t \mid X_{t+u} \leq b_1^{CH}(t+u) \text{ or } X_{t+u} \geq b_2^{CH}(t+u)\}.$$

Since $b_1^{CH}(t) \leq c_1(t)$ and $b_2^{CH}(t) \geq c_2(t)$, we have

$$(5.49) \quad U^c(t + \tau_b, X_{t+\tau_b}) = G^{CH}(t + \tau_b, X_{t+\tau_b}) = V^{CH}(t + \tau_b, X_{t+\tau_b}).$$

By the martingale in equation (5.20) and Optional Sampling Theorem, we have

$$(5.50) \quad \begin{aligned} E_{t,x} e^{-r\tau_b} U^c(t + \tau_b, X_{t+\tau_b}) &= U^c(t, x) + E_{t,x} \int_0^{\tau_b} e^{-ru} (-rL + \delta X_{t+u}) I(X_{t+u} \leq c_1(t+u)) du \\ &+ E_{t,x} \int_0^{\tau_b} e^{-ru} (\delta X_{t+u} - rK) I(X_{t+u} \geq c_2(t+u)) du. \end{aligned}$$

Substituting s by τ_b in (5.37) and taking expectation from both sides, we obtain

$$(5.51) \quad E_{t,x} e^{-r\tau_b} V^{CH}(t + \tau_b, X_{t+\tau_b}) = V^{CH}(t, x).$$

Since part **2** derives $V^{CH}(t, x) \geq U^c(t, x)$ and adhere to equations (5.49), (5.50), and (5.51) in part **5**, we have

$$(5.52) \quad \begin{aligned} E_{t,x} \int_0^{\tau_b} e^{-ru} (-rL + \delta X_{t+u}) I(X_{t+u} \leq c_1(t+u)) du \\ + E_{t,x} \int_0^{\tau_b} e^{-ru} (\delta X_{t+u} - rK) I(X_{t+u} \geq c_2(t+u)) du \geq 0. \end{aligned}$$

Since $x \in (b_1^{CH}(t), c_1(t))$ and the continuity of $b_1^{CH}(t)$ and $c_1(t)$, we have $P_{t,x}(\tau_b > 0) = 1$. By $P_{t,x}(X_{t+u} = c_1(t+u)) = 0$ for $u \in [0, T_1 - t]$, inequality (5.52) can derive the following inequality

$$E_{t,x} \int_0^{\tau_b} e^{-ru} (-rL + \delta X_{t+u}) I(X_{t+u} < c_1(t+u)) du$$

$$(5.53) \quad + E_{t,x} \int_0^{\tau_b} e^{-ru} (\delta X_{t+u} - rK) I(X_{t+u} > c_2(t+u)) du \geq 0.$$

Since $(-rL + \delta X_{t+u})I(X_{t+u} < c_1(t+u)) < 0$ and $(\delta X_{t+u} - rK)I(X_{t+u} > c_2(t+u)) < 0$ for $u \in [0, \tau_b]$, so

$$(5.54) \quad \begin{aligned} & E_{t,x} \int_0^{\tau_b} e^{-ru} (-rL + \delta X_{t+u}) I(X_{t+u} < c_1(t+u)) du \\ & + E_{t,x} \int_0^{\tau_b} e^{-ru} (\delta X_{t+u} - rK) I(X_{t+u} > c_2(t+u)) du < 0, \end{aligned}$$

which is contradictory to (5.53). Hence this disproves our assumption. We finally prove that $c_1(t) = b_1^{CH}(t)$ for $t \in [0, T_1]$. In the same way, we can prove that $c_2(t) = b_2^{CH}(t)$ for $t \in [0, T_1]$. □

6. Comparison To American Strangle Options

An alternative derivative to American chooser options, which also can be used for hedging out high volatility, is the American strangle option (see Qiu (2014)). The American strangle option also has two free boundaries. One may therefore pose the question as to how one decides to buy either the chooser option or strangle option? In particular, under what conditions is it suitable for the investor to buy the American chooser option? To answer these question, we will analyse the returns of American chooser options in two markets (illiquid market and liquid market). By the skeleton analysis (see Peskir and Samee (2011)), we give the region where the returns of American chooser options is better than the American strangle options.

For an illiquid market (such as the over the counter market) the owner of options can not sell the options, but will have the right to exercise them before maturity. At time t and stock price x , the returns in illiquid market is defined as

$$(6.1) \quad R(t, x) = \frac{G(t, x)}{V(0, \frac{L+K}{2})},$$

where $V(0, \frac{L+K}{2})$ is initial price for buying options at time 0 and stock price $\frac{L+K}{2}$, and $G(t, x)$ is the options payoff. At each point (t, x) , we calculate the returns of American chooser options and the returns of American strangle options. After comparing the returns of these two options at each point, we plot Figure 2.

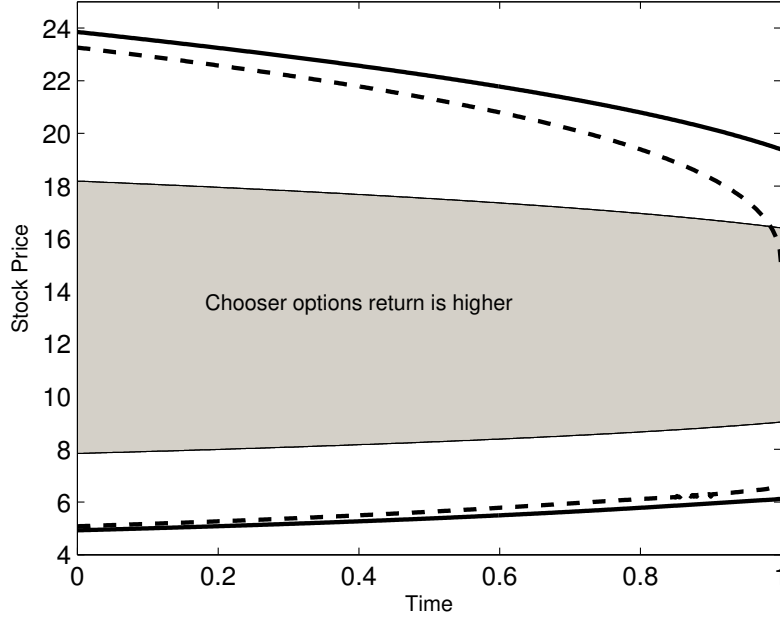


Figure 2. Graph comparing the returns of American chooser and strangle options (in illiquid markets). The solid lines represent the free boundaries of American chooser options and the dash lines represent the free boundaries of American strangle options. The parameter for the American chooser option is $r = 0.04$, $\sigma = 0.3$, $\delta = 0.06$, $L = 10$, $K = 15$, $T_1 = 1$ and $T_2 = T_{call} = T_{put} = 1.2$. The parameter for the American strangle option is $r = 0.04$, $\sigma = 0.3$, $\delta = 0.06$, $L = 10$, $K = 15$ and $T = 1$.

In Figure 2 the shadow region depicts the region where the returns of American chooser options are higher than the returns of American strangle options. If the shadow region can cover the free boundaries of American chooser options, the American chooser options can replace the strangle options for the (highly volatile) asset. However, the following Theorem shows that the shadow region is always inside the continuation region of American chooser options.

Theorem 9. *Let the American chooser options and American strangle options have the same parameters, and the maturity T for strangle options is the same as the chooser maturity T_1 (for chooser options). If $(t, x) \in \bar{D}^{CH}$, we have $R^{ST}(t, x) > R^{CH}(t, x)$, where R^{ST} is the return of American strangle options and R^{CH} is the return of American chooser options.*

Proof. Without loss of generality, we assume that $(t, x) \in \bar{D}_1^{CH}$. By Theorem 2, we know that

$$G^{ST}(x) = G^{CH}(t, x) = L - x.$$

On the other hand, since $T = T_1$, the initial value of American chooser options $V^{CH}(0, \frac{K+L}{2})$ is larger than the initial value of American strangle options $V^{ST}(0, \frac{K+L}{2})$. By the definition of the option returns in (6.1), it is obvious to conclude that

$$R^{CH}(t, x) = \frac{G^{CH}(t, x)}{V^{CH}\left(0, \frac{K+L}{2}\right)},$$

is smaller than

$$R^{ST}(t, x) = \frac{G^{ST}(x)}{V^{ST}(0, \frac{K+L}{2})},$$

for $(t, x) \in \bar{D}^{CH}$. □

From the statement of Theorem 9, if we exercise the American chooser options rationally (we exercise the chooser option when the stock price is inside the optimal stopping region) before T_1 , the returns of American chooser options is always lower than the returns of American strangle options. So an investor aiming to profit from an underlying asset with high volatility, it is reasonable to assume one would buy the American strangle options rather than the chooser options.

We now examine returns in a liquid market. We define a liquid market to mean the owner of options can sell them at anytime before maturity. Since the value of an option is higher than the payoff from exercising an option, the option holder prefers to sell the option rather than exercising the option in liquid markets. In the liquid market, the returns at (t, x) is defined as

$$(6.2) \quad R(t, x) = \frac{V(t, x)}{V(0, \frac{K+L}{2})}.$$

We use $V(t, x)$ to replace $G(t, x)$ in equation (6.1). The Figure 3 shows the region where the returns of American chooser options is higher than the returns of American strangle options.

The shadow region in Figure 3 depicts the region where returns of American chooser options are higher than the returns of American strangle options. If we compare the shadow region in Figures 2 and 3, we observe that the shadow region is smaller in liquid markets. So if the American chooser or strangle option is exercised rationally, it is suitable to select American strangle options for the underlying asset with serious fluctuations.

If we compare Figures 2 and 3, the returns of chooser options are higher than the American strangle options for stock prices 8 to 17 , at T_1 . If the chooser option is not rationally exercised before T_1 , the returns of American chooser options may outperform the American strangle option at the choosing maturity. So we can treat the time period before T_1 as the observing time for the investor to predict the stock price movement in the future rather than making profit from high volatility.

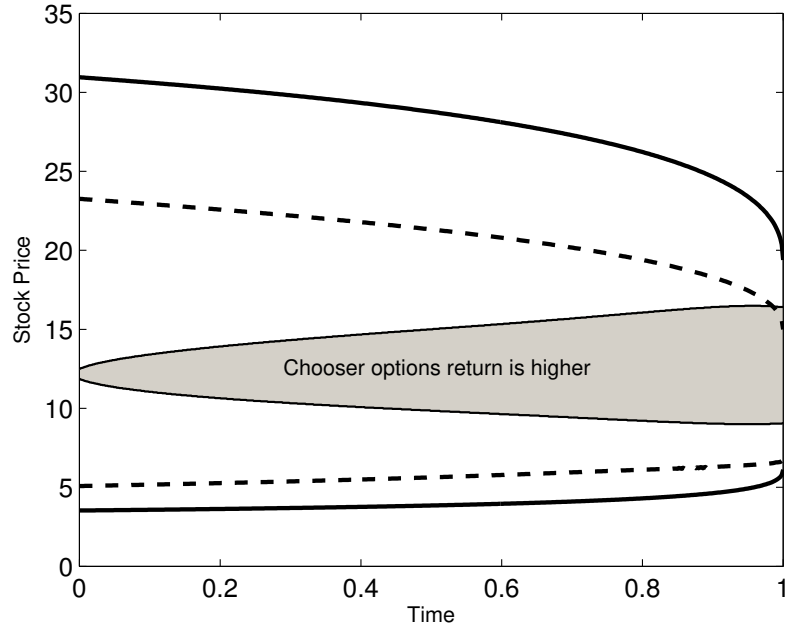


Figure 3. Graph comparing the returns of American chooser and strangle options (in liquid markets). The solid lines represent the free boundaries of American chooser options and the dash lines represent the free boundaries of American strangle options. The parameter for the American chooser option is $r = 0.04$, $\sigma = 0.3$, $\delta = 0.06$, $L = 10$, $K = 15$, $T_1 = 1$ and $T_2 = T_{call} = T_{put} = 1.2$. The parameter for the American strangle option is $r = 0.04$, $\sigma = 0.3$, $\delta = 0.06$, $L = 10$, $K = 15$ and $T = 1$.

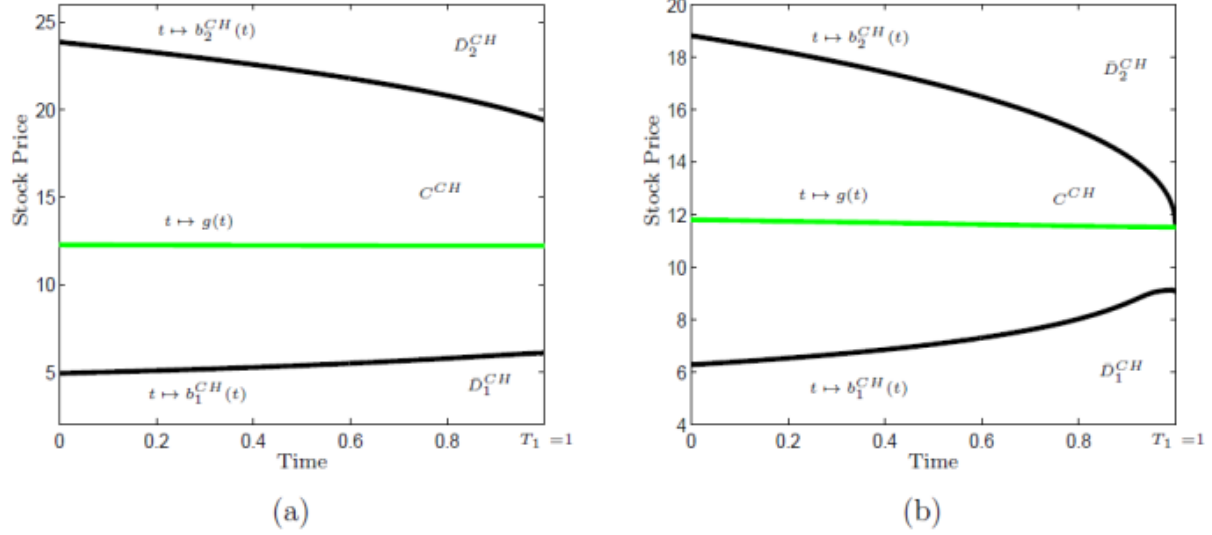


Figure 4. Figure for the two free boundaries for American chooser options. The function $t \mapsto b_2^{CH}(t)$ is the upper free boundary and the function $t \mapsto b_1^{CH}(t)$ is the lower free boundary. The function $g(t)$ is the intersection between the value of American call options and the value of American put options (see (3.6)). The region between b_1^{CH} and b_2^{CH} is the continuation region C^{CH} . The region above b_2^{CH} is the optimal stopping region for chooser American call options and the region below b_1^{CH} is the optimal stopping region to choose American put options. The parameters are $r = 0.04$, $\sigma = 0.3$, $\delta = 0.06$, $T_1 = 1$ and $T_2 = T_{call} = T_{put} = 1.2$. The strike prices used in Figure (a) are $L = 10$ and $K = 15$. The strike prices used in Figure (b) are $L = 15$ and $K = 8$.

7. Numerical Experiments For American Chooser Options

In this section, we show the numerical results of American chooser options, specifically the price, free boundaries, Delta and Gamma. This paper uses the numerical method in Qiu (2014) to calculate the free boundaries of American chooser options and give the options value, Delta and Gamma. Note that previous research on American type options (see Kallast and Kivinukk (2003)) gives the numerical method to price the EEP representation of American put and call options, however this is for one free-boundary problem.

The two free boundaries for chooser options are shown in Figure 4. We see that free boundaries do not intersect and the function $t \mapsto g(t)$ is inside the continuation region, especially when $L > K$ (Figure 4, (b)). Before the choosing maturity T_1 , when the stock price hits the upper boundary, it is optimal to select the American call option immediately. On the other hand, if the stock price hits the lower free boundary, it is optimal to select American put options immediately. After the free boundaries of American chooser options are obtained, we insert them into the equation (5.16). By the Simpson method we numerically estimate the value of the integral and obtain the value of American chooser options.

Stock Price	Price (American)	Price (European)	EEP	Delta Value	Gamma Value
4	6.000	5.842	0.158	-1	0
7	3.207	3.191	0.016	-0.788	0.126
10	1.516	1.514	0.002	-0.314	0.175
13	1.334	1.324	0.009	0.178	0.146
15	1.960	1.930	0.030	0.438	0.114
19	4.471	4.313	0.158	0.782	0.062
22	7.060	6.689	0.370	0.933	0.040
25	10.000	9.305	0.695	1	0

Table 1. Calculations for the American chooser option prices, Delta value and Gamma value, EEP, the price of European chooser options at time $t = 0$, with initial stock prices from 4 to 25. The parameters are $K = 15$, $L = 10$, $r = 0.04$, $\sigma = 0.3$, $\delta = 0.06$, $T_1 = 1$ and $T_2 = T_{call} = T_{put} = 1.2$.

Stock Price	Price (American)	Price (European)	EEP	Delta Value	Gamma Value
4	6.000	5.920	0.080	-1	0
7	3.085	3.080	0.005	-0.884	0.100
10	1.088	1.087	0.001	-0.396	0.199
13	0.778	0.777	0.001	0.175	0.171
15	1.444	1.437	0.007	0.478	0.132
19	4.207	4.130	0.077	0.855	0.062
22	7.001	6.767	0.234	0.994	0.031
25	10.001	9.592	0.409	1	0

Table 2. Calculations for the American chooser option prices, Delta value, Gamma value, EEP, the price of European chooser options at time $t = 0.5$, with initial stock prices from 4 to 25. The parameters are $K = 15$, $L = 10$, $r = 0.04$, $\sigma = 0.3$, $\delta = 0.06$, $T_1 = 1$ and $T_2 = T_{call} = T_{put} = 1.2$.

The value of American chooser options is equal to the value of European chooser options plus the early exercise premium (EEP). The Tables 1 and 2 show the value of chooser options at time $t = 0$ and $t = 0.5$. The value of American chooser options in Figure 5(a) is U-shaped, and the chooser option is more valuable when the stock price is lower than the strike price L or higher than the strike price K . The value of American chooser options is mainly arising from the value of the European chooser option; the EEP is not significant when the stock price is inside $[L, K]$. However when the stock price is approaching and over the the free boundaries of American chooser options, the EEP increases significantly. The reason is that for stock prices approaching the free boundaries, the chooser option has higher opportunity to exercise before the maturity T_1 . If we compare this with European options, the higher probability to exercise before maturity means that higher EEP needs to be paid by the American option buyer.

If we compare the American chooser column and Delta column in Tables 1 and 2, the price of American chooser options is decreasing with t and the Delta is increasing with t . This means that on approaching maturity, the options seller needs to buy or sell more shares to

hedge the option. For the values in the Gamma column, we see that the Gamma is large in the range $[L, K]$ and so the options seller will adjust the Delta hedging portfolio frequently when the stock price is around the region between upper and lower strike prices.

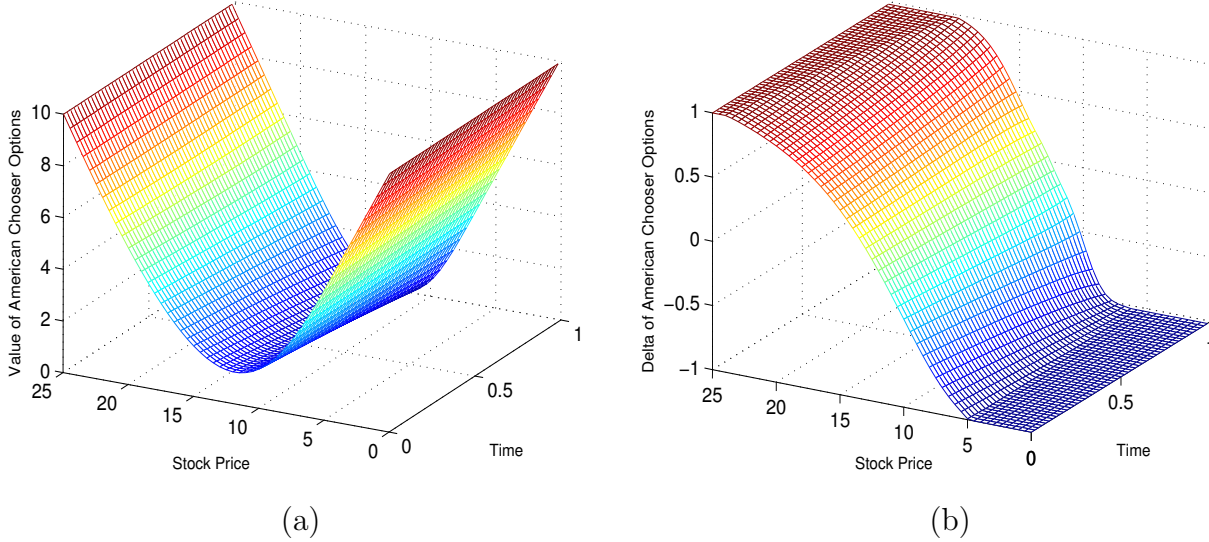


Figure 5. The price and Delta of American chooser options for the variables t and x . The parameters used in the figure: $r = 0.04$, $\sigma = 0.3$, $\delta = 0.06$, $L = 10$, $K = 15$, $T_1 = 1$ and $T_2 = T_{call} = T_{put} = 1.2$.

To see the shape of Delta and Gamma more intuitively, we plot the 3D picture for the two variables t and x . The Figure 5(b) shows that Delta is the continuous and increasing function for stock price when t is given and fixed. The shape looks similarly to the Delta of American strangle options in Qiu (2014). The Gamma represents the sensitivity of Delta to the change of stock price. For the value of Gamma in Figure 6(a), the value is less than 0.3, which is significantly small compared to the Gamma of American strangle options (see Qiu (2014)). For Delta hedging, the option seller does not need to change the hedging strategy as frequently as American strangle options.

Approaching the choosing maturity T_1 , the Gamma of American chooser options in Figure 6(a) has three peaks around the lower strike price L , higher strike price K and the point $g(t)$. After further analysis, we find that the peak at $g(t)$ is contributed to mainly by the European chooser options. If we recall the payoff function of chooser options in (2.7), then it is non-differential at $g(t)$. At time approaching T_1 , the value of American chooser options is close to the value of the chooser payoff. Since $G_x^{CH}(t, g(t)+) > 0$ and $G_x^{CH}(t, g(t)-) < 0$, the value of $x \mapsto G_x^{CH}(t, x)$ at $g(t)$ is extremely sensitive to the change of x . So the Gamma of the chooser option is bigger around $g(t)$ as time approaches the choosing maturity T_1 . Finally, the Gamma of American chooser options in Figure 6(b) has a jump at the free boundary $t \mapsto b_2^{CH}(t)$. So the value function $(t, x) \mapsto V^{CH}(t, x)$ is not the C^2 function in its domain $[0, T_1] \times (0, \infty)$.

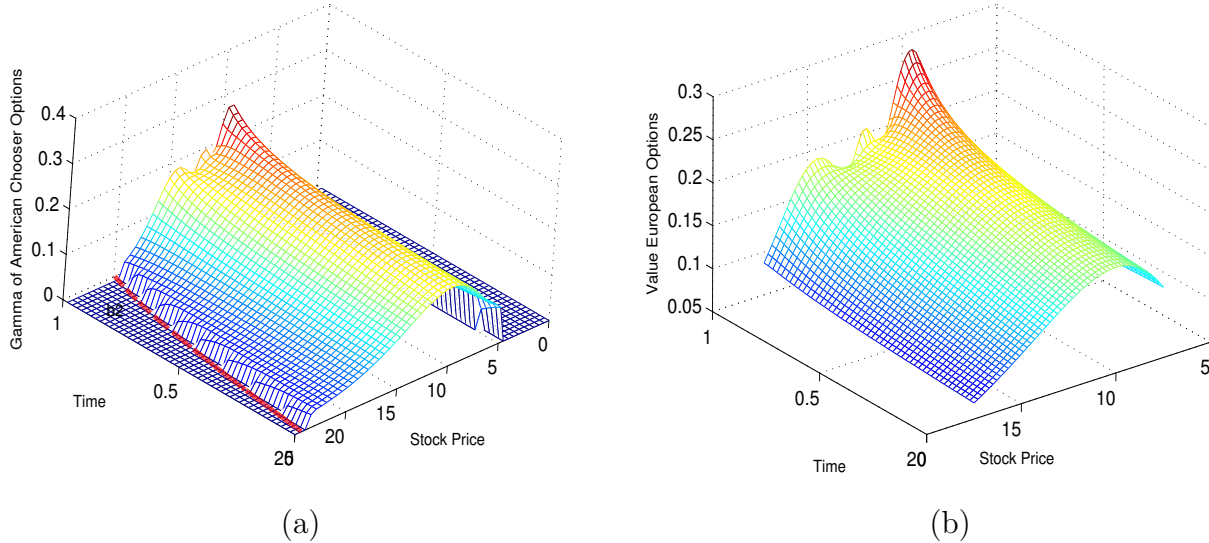


Figure 6. The Gamma of American and European chooser options for the variables t and x . The red line $b_2^{CH}(t)$ in (a) is the upper free boundary of American chooser options. The parameters used in the figure: $r = 0.04$, $\sigma = 0.3$, $\delta = 0.06$, $L = 10$, $K = 15$, $T_1 = 1$ and $T_2 = T_{call} = T_{put} = 1.2$.

8. Conclusion

In this paper we have investigated the mathematical properties of American chooser options. We derived several new Theorems and conducted numerical experiments to illustrate our results. We showed that the two optimal stopping boundaries for American chooser options can be characterised as the unique solution pair to a system formed by two nonlinear integral equations, arising from the early exercise premium representation. We used the early exercise premium representation based on the method of change-of-variable formula with local time on curves. We derived a number of mathematical properties of American chooser options, including smooth-fit, continuity of value function and continuity of free-boundary. We compared American chooser options to American strangle options and analysed our results.

In future work, we would like to investigate the impact of transaction costs upon American chooser option pricing. An extensive empirical analysis of chooser options using asset data before, during and after volatile periods (such as during the financial crisis) may be particularly informative. We would also like to take into account stochastic interest rates in pricing these options, since interest rates typically fluctuate during the life of long-dated options. Finally we would like to extend the American chooser options to include stochastic volatility models on the underlying asset, so that we can more accurately evaluate the option's price based on highly volatile assets.

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